Quantum Fisher - Bures information of two-level systems and a three-level extension

1996 J. Phys. A: Math. Gen. 29 L271

(http://iopscience.iop.org/0305-4470/29/10/008)

J. Phys. A: Math. Gen. **29** (1996) L271–L275. Printed in the UK

LETTER TO THE EDITOR

Quantum Fisher–Bures information of two-level systems and a three-level extension

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Received 17 November 1995

Abstract. Braunstein and Caves have recently demonstrated that the Bures metric on the mixed quantum states is equivalent—up to a proportionality factor of four—to the statistical distinguishability or quantum Fisher information metric. The volume element of these metrics can then—adapting a fundamental Bayesian principle of Jeffreys to the quantum context—serve as a reparametrization-invariant prior measure over the quantum states. The implications of this line of reasoning for the two-level systems, in general, and an embedding of them into a certain set of three-level systems are investigated.

In this letter, we study, among other topics, the four-dimensional convex set of three-level (spin-1) density matrices of the particular form ($0 \leqslant v \leqslant 1$)

$$\rho = \frac{1}{2}\begin{pmatrix} v+z & 0 & x-\mathrm{i}y \\ 0 & 2-2v & 0 \\ x+\mathrm{i}y & 0 & v-z \end{pmatrix}. \tag{1}$$

For $v = 1$, the middle level is inaccessible and, in effect, the two-level (spin-$\frac{1}{2}$) density matrices

$$\rho = \frac{1}{2}\begin{pmatrix} 1+z & x-\mathrm{i}y \\ x+\mathrm{i}y & 1-z \end{pmatrix} \tag{2}$$

are recovered. The domain of admissible (due to trace and non-negativity constraints) values of the parameters x, y and z is then the unit ball (Bloch or Poincaré sphere), $x^2+y^2+z^2 \leqslant 1$. So, equation (1) serves as one of many possible extensions or generalizations of (2) (cf [1]). (Physical photons, although spin-1 particles, are, due to their masslessness, describable by (2).)

Let us attach to the domain of 2×2 density matrices (2), the Bures metric given by [2, formula (3.7)], cf [3, 4]

$$\frac{1}{4}\,\mathrm{Tr}\left\{\mathrm{d}\rho\,\mathrm{d}\rho + \frac{1}{|\rho|}(\mathrm{d}\rho - \rho\,\mathrm{d}\rho)(\mathrm{d}\rho - \rho\,\mathrm{d}\rho)\right\}. \tag{3}$$

The matrix (g_{ij}; $i, j = x, y, z$) corresponding to this metric is

$$\frac{1}{4(1-x^2-y^2-z^2)}\begin{pmatrix} 1-y^2-z^2 & xy & xz \\ xy & 1-x^2-z^2 & yz \\ xz & yz & 1-x^2-y^2 \end{pmatrix}. \tag{4}$$

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0305-4470/96/100271+05$19.50

Its inverse (g^{ij}) assumes the simple form

$$4\begin{pmatrix} 1-x^2 & -xy & -xz \\ -xy & 1-y^2 & -yz \\ -xz & -yz & 1-z^2 \end{pmatrix}. \tag{5}$$

The associated volume element [5] of the Bures metric is obtained by taking the square root of the determinant of (4). The result is

$$1/8(1-x^2-y^2-z^2)^{1/2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,. \tag{6}$$

Braunstein and Caves [6] have shown that the Bures metric (which extends to mixed states the Fubini–Study metric on pure states) is simply proportional (by a factor of four) to the Fisher information (statistical distinguishability) metric on the quantum states. Relying upon this essential equivalence, together with Jeffreys' principle [7–9] for using the square root of the determinant of the (classical) Fisher information matrix as a reparametrization-invariant prior, we advance (6) as a prior measure over the two-level density matrices (2). From it—through a normalization—one obtains a prior probability distribution

$$p(x,y,z) = 1/\pi^2(1-x^2-y^2-z^2)^{1/2} \tag{7}$$

over the unit ball, $x^2+y^2+z^2 \leqslant 1$. (The average of the von Neumann entropy, $-\operatorname{Tr}\rho\ln\rho$, over the unit ball is then $2\ln 2 - 7/6 \approx 0.219\,627\,7$, cf [10].)

Since

$$\int_{-(1-x^2-y^2)^{1/2}}^{(1-x^2-y^2)^{1/2}} p(x,y,z)\,\mathrm{d}z = 1/\pi \tag{8}$$

the (three) bivariate marginal probabilities of (7) are uniform distributions over unit discs ($x^2+y^2 \leqslant 1,\ldots$)—agreeing, in this particular manner, with Laplace's principle of insufficient reason [11]. Then, the three univariate marginal distributions are of the form

$$\int_{-(1-x^2)^{1/2}}^{(1-x^2)^{1/2}} 1/\pi\,\mathrm{d}y = 2(1-x^2)^{1/2}/\pi \qquad (-1 \leqslant x \leqslant 1)\,. \tag{9}$$

Under the transformation, $x = 2q-1$, this becomes a beta distribution (figure 1),

$$8q^{1/2}(1-q)^{1/2}/\pi \qquad (0 \leqslant q \leqslant 1) \tag{10}$$

with its two parameters equalling $\frac{3}{2}$. The family of beta distributions is typically employed for the role of prior distributions over binomial $(0/1)$ parameters [9].

As an illustration of the application of Bayes' theorem [8, 9] to the estimation of quantum systems [12–14], let us hypothesize an experimental situation in which spin measurements are performed on each of fourteen replicas of a two-level quantum system—three are taken in the X-direction with two 'ups' recorded, five in the Y-direction with three 'ups' and six in the Z-direction with two 'ups'. Then the posterior (modified) probability distribution over the unit ball is proportional to the product of the prior (7) and the likelihood

$$\left(\frac{1-x}{2}\right)\left(\frac{1+x}{2}\right)^2\left(\frac{1-y}{2}\right)^2\left(\frac{1+y}{2}\right)^3\left(\frac{1-z}{2}\right)^4\left(\frac{1+z}{2}\right)^2 \tag{11}$$

since in a two-level system with parameters x, y, z, the probability of an 'up' in the X-direction is $(1+x)/2$ and a 'down', $(1-x)/2, \ldots$. This product can be normalized, through an integration over the unit ball, to comprise the posterior probability distribution

$$7168(1-x)(1+x)^2(1-y)^2(1+y)^3(1-z)^4(1+z)^2/1903\pi^2(1-x^2-y^2-z^2)^{1/2}\,. \tag{12}$$

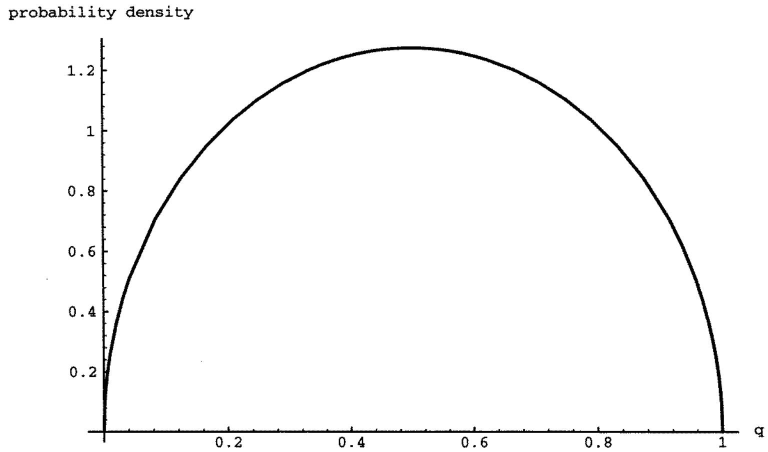

Figure 1. Univariate marginal probability distribution for two-level systems, expressed as a beta distribution (10).

Let us now attempt to extend this line of analysis to the three-level density matrices of the form (1). The Bures metric for such systems is given by [2, formula (3.8)]

$$\frac{1}{4}\,\mathrm{Tr}\left\{\mathrm{d}\rho\,\mathrm{d}\rho+\frac{3}{1-\mathrm{Tr}\,\rho^3}(\mathrm{d}\rho-\rho\,\mathrm{d}\rho)(\mathrm{d}\rho-\rho\,\mathrm{d}\rho)+\frac{3|\rho|}{1-\mathrm{Tr}\,\rho^3}(\mathrm{d}\rho-\rho^{-1}\,\mathrm{d}\rho)(\mathrm{d}\rho-\rho^{-1}\,\mathrm{d}\rho\right\}. \tag{13}$$

The result is representable by the matrix $(g_{ij};\, i,\, j = v, x, y, z)$

$$\frac{1}{4(v^2-x^2-y^2-z^2)} \times \begin{pmatrix} (x^2+y^2+z^2-v)/(1-v) & -x & -y & -z \\ -x & (y^2+z^2-v^2)/v & xy/v & xz/v \\ -y & xy/v & (x^2+z^2-v^2)/v & yz/v \\ -z & xz/v & yz/v & (x^2+y^2-v^2)/v \end{pmatrix} \tag{14}$$

having the particularly simple inverse $(g^{ij};\, i,\, j = v, x, y, z)$

$$4\begin{pmatrix} (1-v)v & (1-v)x & (1-v)y & (1-v)z \\ (1-v)x & v-x^2 & -xy & -xz \\ (1-v)y & -xy & v-y^2 & -yz \\ (1-v)z & -xz & -yz & v-z^2 \end{pmatrix}. \tag{15}$$

(Removing the factor four from (15) yields the inverse of the quantum Fisher information matrix [6, 15–17]. This then serves—in a non-Bayesian application—as a (Cramér–Rao) lower bound, in the sense of non-negative definiteness, on the covariance matrix of unbiased estimates of the parameters, v, x, y, z [15–17].)

The square root of the determinant of (14) is (cf equation (6))

$$1/16v(1-v)^{1/2}(v^2-x^2-y^2-z^2)^{1/2}. \tag{16}$$

This can be normalized, using spherical coordinates (r, θ, ϕ) to perform the integrations, to

$$3/4\pi^2 v(1-v)^{1/2}(v^2-x^2-y^2-z^2)^{1/2}. \tag{17}$$

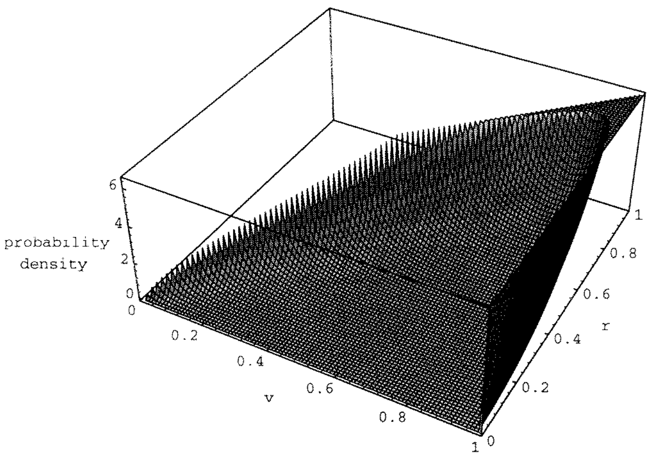

Figure 2. Bivariate marginal probability distribution over auxiliary parameter (v) and radial parameter (r) for the extended system.

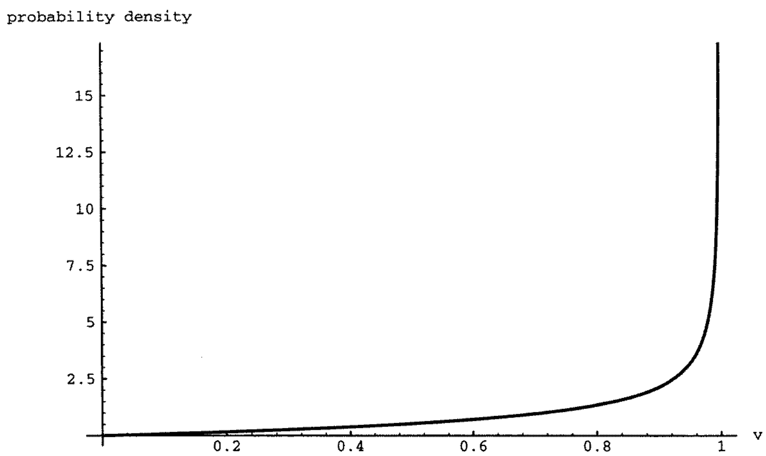

Figure 3. Univariate marginal (β) probability distribution (19) over auxiliary parameter (v) for extended system.

(The ranges, $0 \leqslant r = (x^2 + y^2 + z^2)^{1/2} \leqslant v$ and $0 \leqslant v \leqslant 1$ were employed.) In spherical coordinates, equation (17) assumes the form

$$3r^2 \sin\theta / 4\pi^2 v(1-v)^{1/2}(v^2 - r^2)^{1/2}\,. \tag{18}$$

(Figure 2 shows the marginal distribution of (18) over r and v.) The univariate marginal distribution of (17) and (18) over the variable v is, again (cf equation (10)), an (asymmetric) beta distribution (figure 3),

$$3v(1-v)^{-1/2}/4 \qquad (0 \leqslant v \leqslant 1) \tag{19}$$

with its two parameters equalling 2 and $\frac{1}{2}$. Holding v fixed (V), the conditional distributions of (17) and (18) are

$$1/\pi^2 V^2 (V^2 - x^2 - y^2 - z^2)^{1/2} \tag{20}$$

$$r^2 \sin\theta / \pi^2 V^2 (V^2 - r^2)^{1/2}\,. \tag{21}$$

(For $V = 1$, equation (20) reduces to (7).) However, integrations could not be exactly nor numerically performed over v to obtain the corresponding marginal distributions over r, θ, ϕ or x, y, z.

I would like to express appreciation to the Institute for Theoretical Physics for computational support in this research.

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